Probability and Measure

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BSc Applied Mathematics

April 2023-June 2023

Contents

1 Acknowledge

This course notes are made for the course Probability & Measure (WBMA024-05) of the University of Groningen during the academic year 2022-2023. The course is held by prof. dr. J.P. (Pieter) Trapman, and followed the lecture notes made by de Snoo and Winkler, n.d. This document is a summary of the course and do not substitute the corresponding lecture notes.

To complete this course notes, I have also used Williams, 1991 and Durrett, 2019.

2 Introduction

Measure and integration offers a general approach to the theory of integration based on measure theory. The starting point is an abstract framework through the study of collections of sets having desirable properties (the so-called sigma-algebras), and of real-valued functions defined on these collections (the measures themselves). From this, one defines measurable functions, integrable functions and the Lebesgue integral. This is a far reaching extension of the well known theory of Riemann integration. The approach via measures provides a wide variety of applications: in particular about the interchange of various limiting procedures. Also, a connection with functional analysis is provided via the introduction of spaces of Lebesgue integrable functions. The course explores connections with probability theory and includes the mathematical concepts needed to understand stochastic processes.

3 Algebras and Measures

We begin with a formal definition of what a measure space is. The idea is that we will have a triple $(\Omega, \mathcal{A}, \mu)$ where Ω is the space, $\mathcal{A}$ is particular collection of subset of Ω and μ is he 'measure', hence a function used to compute the measure, i.e, the 'size/volume', of a set.

Definition 1 ***Algebra*** *A collection $\mathcal{A}$ of subsets of a set Ω is an algebra (or field) if*

(1) $\Omega \in \mathcal{A}$;

(2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$;

(3) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$.

Note that in algebra one has

1. $\emptyset = \Omega^c \in \mathcal{A}$;
2. $A, B \in \mathcal{A} \Rightarrow A \cap B = (A^c \cup B^c)^c \in \mathcal{A}$;
3. $A, B \in \mathcal{A} \Rightarrow A \backslash B = A \cap B^c \in \mathcal{A}$

Definition 2 ***Measure*** *A finitely additive measure μ on an algebra $\mathcal{A}$ us an extended real-valued function $\mu : \mathcal{A} \to [0, \infty]$ which satisfies*

1. $\mu(\emptyset) = 0$;

2. *$A, B \in \mathcal{A}$ pairwise disjoint $\Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$*

Definition 3 ***σ-algebra*** *A collection of $\mathcal{A}$ of subset of a set Ω is a σ-algebra (or σ-field) if*

1. *$\Omega \in \mathcal{A}$;*
2. *$A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$;*
3. *$A_n \in \mathcal{A}$, $n \in \mathbb{N} \Rightarrow \cup_{n=1}^{\infty} A_n \in \mathcal{A}$*

Example: The collection $\mathcal{A} = \{\emptyset, \Omega$ is the smallest σ-algebra with respect to Ω, and the power set is larger σ-algebra with respect to Ω.

Definition 4 ***Measure on an σ-algebra*** *A finitely additive measure μ on an algebra $\mathcal{A}$ us an extended real-valued function $\mu : \mathcal{A} \to [0, \infty]$ which satisfies*

1. *$\mu(\emptyset) = 0$;*
2. *$A_n \in \mathcal{A}$ pairwise disjoint $\Rightarrow \mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$*

Remark: Let $A_n \in \Omega$, $n \in \mathbb{N}$, and define the subsets

$$A'_1 = A_1, \quad A'_n = A_n \backslash (A_1 \cup \cdots \cup A_{n-1}), \quad n \geq 2$$

Then the sets A'_n are pairwise disjoint and their countable union is equalt to the corresponding countable union of A_n.

Definition 5 ***Measurable space*** *A measurable space (Ω,$\mathcal{A}$) a set Ω provided with a σ-algebra $\mathcal{A}$ on Ω. The elements of $\mathcal{A}$ are called measurable sets.*

Definition 6 ***Measure space*** *A measure space is a triple $(\Omega, \mathcal{A}, \mu)$ consisting of a set Ω, a σ-algebra $\mathcal{A}$ on Ω, and a measure μ. The space is finite if $\mu(\Omega) < \infty$.*

Definition 7 ***Probability space*** *A probability space is a measure space with $\mu(\Omega) = 1$.*

3.1 Generators of σ-algebra

There are multiple ways to construct σ-algebra on a set Ω. A typical way is to define a σ-algebra given a collection of subset.

Theorem 8 *The intersection of a nonempty family of σ-algebra on a set Ω is a σ-algebra.*

Proposition 9 *Let D be a collection of subset of a set Ω. Then there is precisely one σ-algebra $\mathcal{A}$ such that*

- *$D \in \mathcal{A}$;*
- *if $\mathcal{B}$ is a σ-algebra with $D \subset \mathcal{B}$, then $\mathcal{A} \subset \mathcal{B}$*

Definition 10 ***Generator*** *Let D be a collection of subset of a set Ω. The unique σ-algebra in the previous proposition is said to be generated by D, denoted by $\sigma(D)$, and D is said to be the generator of this σ-algebra.*

Remark: A σ-algebra $\mathcal{A}$ can be generated by different collection of subsets of Ω.

3.1.1 Borel σ-algebra

Definition 11 *Let Ω be a topological space. The σ-algebra generated by all open sets of Ω is the Borel σ-algebra $\mathcal{B}(\Omega)$.*

Proposition 12 *The Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$ is generated by*

- *the collection of closed subsets in $\mathbb{R}$;*
- *the collection of intervals $(-\infty, b]$ $b \in \mathbb{R}$;*
- *the collection of intervals $(a, b]$, $a, b \in \mathbb{R}$ and $a < b$.*

Moreover, the Borel σ-algebra $\mathcal{B}^d$ on $\mathbb{R}^d$ is generated by

- *the collection of closed subsets;*
- *the collection of half-spaces;*
- *the collection of rectangles*

3.1.2 π-systems and d-systems

Definition 13 ***π-system*** *A π-system, $\mathcal{A}$, is a set containing $\emptyset$ with the property that if $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$.*

Definition 14 ***d-system*** *A d-system, or Dynkin system, $\mathcal{A}$, is a collection of subsets of Ω such that*

- $\Omega \in \mathcal{A}$;
- $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$;
- $A_n \in \mathcal{A}$, $n \in \mathbb{N}$ *pairwise disjoint* $\Rightarrow \cup_{n=1}^{\infty} A_n \in \mathcal{A}$

Remark: A σ-algebra $\mathcal{A}$ is a d-system which is closed under intersections (It holds in both directions).

3.2 Completion of σ-algebra

Definition 15 *A measure space $(\Omega, \mathcal{A}, \mu)$ is complete if every subset of a set of measure 0 is measurable (and, hence, is itself a set of measure 0)*

Definition 16 *A measure space $(\Omega, \mathcal{B}, \nu)$ is said to* extend *a measure $(\Omega, \mathcal{A}, \mu)$ if $\mathcal{A} \subset \mathcal{B}$ and $\nu(E) = \mu(E)$ for all $E \in \mathcal{A}$.*

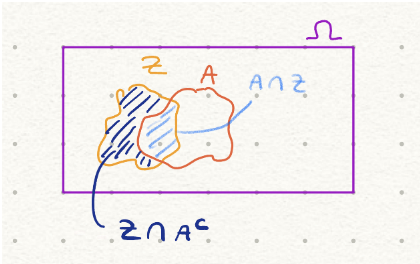
3.3 Outer measures

Definition 17 *Let Ω be a set and let $P(\Omega)$ be its power set. An extended real-valued function $\mu^* : P(\Omega) \to [0, \infty]$ is an outer measure on Ω if*

- $\mu^*(\emptyset) = 0$;
- $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$;
- $A_n \in \Omega,\ n \in \mathbb{N} \Rightarrow \mu^*(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$

Definition 18 *Let μ^* be an outer measure on Ω. A set $A \subset \Omega$ is measurable with respect to μ^* if for any set $Z \subset \Omega$*

$$\mu^*(Z) = \mu^*(Z \cap A) + \mu^*(Z \cap A^c)$$



3.4 Lebesgue measure

Lebesgue measure is probably the most famous and fundamental measure. It is a measure on $\mathbb{R}^d$ and is that which corresponds to our intuitive idea of how bit a set is. In 1-D it is defined by

$$\mu([a, b]) = b - a$$

Definition 19 ***Lebesgue measure*** *Let $A \subset \mathbb{R}^d$. Then $\mu(A) \in [0, \infty]$ is defined by*

$$\mu(A) = \inf \left\{ \sum_{n=1}^{\infty} l(R_n) \ :\ R_n \subset \mathbb{R}^d \quad \textit{closed rectangle},\ A \subset \bigcup_{n=1}^{\infty} R_n \right\}$$

, where $l(R)$ is the volume of R.

Remark: The rectangles R_n, $n \in \mathbb{N}$, form a cover of A and $\mu(A)$ is the infimum of the total volumes $\sum_{n=1}^{\infty} l(R_n)$ of all possible covers R_n

Lemma 20 *Let the Lebesgue outer measure μ and let $R \subset \mathbb{R}^d$ be a closed rectangle with volume $l(R)$. Then*

$$\mu(R) = l(R)$$

Proposition 21 *Every closed rectangle in $\mathbb{R}^d$ is Lebesgue measurable.*

Corollary 22 *Every open set in R^d is Lebesgue measurable.*

Corollary 23 *Every Lebesgue measure on $\mathbb{R}^d$ is σ-finite.*

Proposition 24 *Let $A \subset \mathbb{R}^d$ and let $x \in \mathbb{R}^d$. Then the Lebesgue measure μ is invariant under translation.*

3.4.1 Lebesgue-Stieltjes measure

3.5 Measure Theoretic Formulation of Probability

In Probability theory it is common to write the measure space as $(\Omega, \mathcal{A}, \mathbb{P})$ with the additional assumption that $\mathbb{P}(\Omega) = 1$. In this setting Ω is the set of all possible individual outcomes (of the experiment, or random process), and $A \in \mathcal{A}$ are the measurable set called events. For an example suppose we are going to toss a coin twice and we want to look at what results we get. Our set is that of all possible sequences

$$\Omega = \{TT, TH, HT, HH\}.$$

We can simply define the σ-algebra to be $P(\Omega)$. Then we can see that if we want to see the probability of something occuring, for instance at least one head appearing, then this defines a subset of Ω which is in $\mathcal{A}$ and we can find this probability,

$$\mathbb{P}(\{\text{At least one head appears}\}) = \mathbb{P}(\{TH, HT, HH\}).$$

4 Measurability of functions

Measurable functions are structure-preserving function between measurable spaces. An easy example of measurable function are the random variables.

Definition 25 *Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measurable spaces. The mapping $f : \Omega \to \Omega'$ is measurable, or, more precisely, $(\mathcal{A}, \mathcal{A}')$-measurable, if $f^{-1}(A') \in \mathcal{A}$ for any $A' \in \mathcal{A}$*

Remark: if two mappings are measurable, then the composition is also measurable.

Theorem 26 *Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measurable spaces, ane let D be a generator for $\mathcal{A}'$. Then the following statements are equivalent:*

- *the mapping $f : \Omega \to \Omega'$ is measurable;*
- *$f^{-1}(E') \in \mathcal{A}$ for any $E' \in D$*

Proposition 27 *Let Y be a topological space and let $\mathcal{B}(Y)$ be the σ-algebra generated by the open sets in Y. Let f be a mapping from the measurable space $(\Omega, \mathcal{A})$ to the measurable space $(Y, \mathcal{B}(Y))$. Then the following statements are equivalent:*

- *f is measurable;*
- *the set $f^{-1}(O) \in \mathcal{A}$ for each open set $O \in Y$*

Definition 28 *Let f be a function from the measurable space $(\Omega, \mathcal{A})$ to $\mathbb{R}$. Then f is called measurable if f is measurable as a function from $(\Omega, \mathcal{A})$ to $(\mathbb{R}, \mathcal{B})$.*

Remark: The Borel σ-algebra $\mathcal{B} = \mathcal{B}(\mathbb{R})$ is also generated by half-open and half closed intervals.

Corollary 29 *The real-valued function $f : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is measurable if and only if for each $c \in \mathbb{R}$ on (and hence all) of the following assertions are satified:*

- $\{\omega \in \Omega : f(\omega) < c\} \in \mathcal{A}$
- $\{\omega \in \Omega : f(\omega) \leq c\} \in \mathcal{A}$
- $\{\omega \in \Omega : f(\omega) > c\} \in \mathcal{A}$
- $\{\omega \in \Omega : f(\omega) \geq c\} \in \mathcal{A}$

Definition 30 *Let f be a mapping from the measurable space $(\Omega, \mathcal{A})$ to $\mathbb{R}^2$. Then f is called measurable if f is measurable as a mapping from $(\Omega, \mathcal{A})$ to $(\mathbb{R}^2, \mathcal{B}^2)$*

Proposition 31 *Let $f = (f_1, f_2)$ be a mapping from the measurable space $(\Omega, \mathcal{A})$ to $\mathbb{R}^2$. Then f is measurable if and only if the functions f_1 and f_2 are measurable.*

4.1 Measurable functions

Lets denote $\overline{\mathbb{R}}$ and $\overline{\mathcal{B}}$ the extended real line and the the corresponding Borel σ-algebra.

Proposition 32 *Let f be an extended real-valued function from the measurable space $(\Omega, \mathcal{A})$ to $(\overline{\mathbb{R}}, \overline{\mathcal{B}})$. Then the following statements are equivalent*

- *f is measurable;*
- *the set $f^{-1}(O) \in \mathcal{A}$ for each open set $O \subset \overline{\mathbb{R}}$.*

Corollary 33 *The real-valued function $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ is measurable if and only if for each $c \in \mathbb{R}$ on (and hence all) of the following assertions are satified:*

- $\{\omega \in \Omega : f(\omega) < c\} \in \mathcal{A}$
- $\{\omega \in \Omega : f(\omega) \leq c\} \in \mathcal{A}$
- $\{\omega \in \Omega : f(\omega) > c\} \in \mathcal{A}$
- $\{\omega \in \Omega : f(\omega) \geq c\} \in \mathcal{A}$

Proposition 34 *Let the extended measurable real-valued functions $f, g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$, and let $A \in \mathcal{A}$. Then the sets*

$$\{\omega \in \Omega \,:\, f(\omega) < g(\omega)\}, \quad \{\omega \in \Omega \,:\, f(\omega) \leq g(\omega)\}, \quad \{\omega \in \Omega \,:\, f(\omega) = g(\omega)\}$$

are measurable.

Proposition 35 *Let $f, g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable extended real-valued functions. Then the maximum and the minimum of these functions are measurable.*

Proposition 36 *Let $f_n : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be a sequence of measurable extended real-valued functions. Then also*

$$\sup_{n \in \mathbb{N}} f_n, \quad \inf_{n \in \mathbb{N}} f_n, \quad \limsup_{n \in \mathbb{N}} f_n, \quad \liminf_{n \in \mathbb{N}} f_n$$

are measurable extended real-valued functions from $(\Omega, \mathcal{A})$ to $(\overline{\mathbb{R}}, \overline{\mathcal{B}})$.

Definition 37 *Let (x_n) be a sequence in $\overline{\mathbb{R}}$. Then*

$$\limsup_{n\in\mathbb{N}} x_n := \lim_{n\to\infty} (\sup_{r\geq n} x_r),$$

and

$$\liminf_{n\in\mathbb{N}} x_n := \lim_{n\to\infty} (\inf_{r\geq n} x_r).$$

Corollary 38 *Let (x_n) be a sequence in $\overline{\mathbb{R}}$. Then*

$$\lim_{n\to\infty} x_n = l \quad \Leftrightarrow \quad \liminf_{n\in\mathbb{N}} x_n = \limsup_{n\in\mathbb{N}} x_n = l$$

Definition 39 *Let Ω be a set. A sequence of extended real-valued functions $f_n : \Omega \to \overline{\mathbb{R}}$ converges pointwise to an extended real-value function $f : \Omega \to \overline{\mathbb{R}}$ if $f_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.*

Proposition 40 *Let $(\Omega, \mathcal{A})$ be a measurable space. Let the sequence of measurable real-valued functions $f_n : \Omega \to \overline{\mathbb{R}}$ converges pointwise to an extended real-valued function $f : \Omega \to \overline{\mathbb{R}}$. If all f_n are measurable, then f is measurable.*

Proposition 41 *Let $f, g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable extended real-valued functions. Then the sum, multiplication and division among them are also measurable, on their domains of definitions.*

4.2 Random Variables

A random variable is a measurable function from a probability space. For example if we again have the probability space generated by tossing a coin twice. Then if X counts the number of heads, it is a random variable with landing space $\mathbb{N}$ with σ-algebra $P(N)$ often the landing space of a random variable is not made specific. In particular its σ-algebra may not be made explicit. The random variable induces a measure on its landing space

$$\mu_x = \mathbb{P} \circ X^{-1}$$

This is called the law of X.

In a more precise definition,

Definition 42 *If $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space and $X : \Omega \to \overline{\mathbb{R}}$ is measurable, then X is called a random variable. In general, if $X : \Omega \to S$, where $(S, \mathcal{A})$ is a measurable space, we call X a random quantity.*

Theorem 43 ***Induced measure*** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $(S, \mathcal{A})$ be a measurable space. Let $f : \Omega \to S$ be a measurable function. Then f induces a measure on $(S, \mathcal{A})$ defined by $\nu(A) = \mu(f^{-1}(A))$ for each $A \in \mathcal{A}$.*

Definition 44 *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let (S,A) be a measurable space. Let $X : \Omega \to S$ be a random quantity. Then the measure induced on $(S, \mathcal{A})$ from $\mathbb{P}$ by X is called the distribution of X.*

So, let $(\Omega, \mathcal{A}, \mathbb{P})$ and let a random variable X associated with this space. Then,

$$\Omega \xrightarrow{X} \overline{\mathbb{R}}$$
$$[0,1] \xleftarrow{\mathbb{P}} \mathcal{A} \xleftarrow{X^{-1}} \overline{\mathcal{B}}$$
$$[0,1] \xleftarrow{\mathbb{P}} \sigma(\Omega) \xleftarrow{X^{-1}} \overline{\mathcal{B}}$$

The law μ_X of X is defined by

$$\mu_X = \mathbb{P} \circ X^{-1}$$

and the distribution function $F_X : \mathbb{R} \to [0,1]$ is defined as

$$F_X(c) := \mu_X(-\infty, c] = \mathbb{P}(X \leq c) = \mathbb{P}(X^{-1}((-\infty, c])) = \mathbb{P}(\{\omega \,:\, X(\omega) \leq c\})$$

5 Approximation by simple functions

Simple functions are the building blocks of measurable function. Every nonnegative measurable function is the limit of simple function.

Definition 45 *A real-valued function $f : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is called simple if f is measurable and takes on only finite number of values.*

Remark: If $f : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is simple and $\alpha_1, \dots, \alpha_r \in \mathbb{R}$ are its values then the sets

$$A_k = f^{-1}(\alpha_k), \quad k = 1, \dots, r,$$

are measurable and pairwise disjoint. Therefore the function f can be written as

$$f = \sum_{k=1}^{r} \alpha_k \mathbb{1}_{A_k}$$

Theorem 46 *Let the extended real-valued function $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable. Then there exists a sequence of simple functions f_n on Ω such that $f_n \to f$ pointwise. Moreover,*

1. *if f is bounded, then the sequence converge uniformly;*
2. *if $f \geq 0$, then the sequence f_n may be chosen such that*

$$f_n \geq 0, \quad f_n \leq f_{n+1}.$$

5.1 Properties which are valid almost everywhere

Definition 47 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. A property is said to hold almost everywhere (a.e.) on Ω if it holds on a measurable set $A \in \mathcal{A}$ whose complement A^c has measure 0.*

Lemma 48 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let f, g, and h be extended real-valued functions from Ω to $\overline{\mathbb{R}}$, such that $f = g$ a.e. and $g = h$ a.e. Then $f = h$.*

Lemma 49 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let f and g be measurable extended real-valued functions from Ω to $\overline{\mathbb{R}}$. Then*

$$f = g \, a.e. \quad \Leftrightarrow \quad \mu(\{\omega \in \Omega \, : \, f(\omega) \neq g(\omega)\}) = 0.$$

Remark: Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. A sequence of extended real-valued functions f_n from Ω to $\overline{\mathbb{R}}$ converges pointwise almost everywhere to an extended real-valued function f from Ω to $\overline{\mathbb{R}}$, i.e.,

$$\lim_{n\to\infty} f_n = f \, a.e.,$$

if there exists a set $A \in \mathcal{A}$ such that

$$f_n(\omega) \to f(\omega), \quad \omega \in A$$

while $\mu(A^c) = 0$.

Lemma 50 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let f_n and f be measurable extended real-valued functions from Ω to $\overline{\mathbb{R}}$, then*

$$\lim_{n\to\infty} f_n = f \, a.e. \Leftrightarrow \mu(\{\omega \in \Omega \, : \, \lim_{n\to\infty} f_n(\omega) \neq f(\omega)\}) = 0.$$

Theorem 51 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let f and g be extended real-valued functions from Ω to $\overline{\mathbb{R}}$, such that $f = g$ a.e. If f is measurable, then g is measurable.*

Proposition 52 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let f_n and f be extended real valued functions from Ω to $\overline{\mathbb{R}}$. Assume that f_n is measurable. Then*

$$\lim_{n\to\infty} f_n = f \, a.e. \quad \Rightarrow \quad f \text{ is measurable.}$$

6 Integrability of functions

6.1 Integral of nonnegative measurable functions

Definition 53 (4.2) *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ be a nonnegative simple function with values $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$. Define*

$$A_k = f^{-1}(\alpha_k), \quad k = 1, \ldots, r,$$

so that the sets A_k are measurable, pairwise disjoint, and the function f can be written as

$$f = \sum_{k=1}^{r} \alpha_k \mathbb{1}_{A_k}.$$

The integral $\int_\Omega f d\mu$ of f over Ω with respect to μ, is defined by

$$\int_\Omega f d\mu = \sum_{k=1}^{r} \alpha_r \mu(A_k).$$

Remark: The integral $\int_\Omega f\,d\mu$ is often defined in different ways,

$$\int_\Omega f(\omega)d\mu(\omega), \quad \int_\Omega f(\omega)\,\mu(d\omega).$$

This notation will be used when several measures are considered simultaneouslu.

Proposition 54 (4.3) *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $f, g : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ be nonnegative simple functions and $\alpha \geq 0$. Then αf and $f+g$ are also nonnegative simple functions and*

1. $\int \Omega \alpha f d\mu = \alpha \int_\Omega f d\mu$, $\alpha > 0$;
2. $\int_\Omega (f+g) d\mu = \int_\Omega f d\mu + \int_\Omega g d\mu$;
3. $f \leq g \Rightarrow \int_\Omega f d\mu \leq \int_\Omega g d\mu$.

Corollary 55 (4.4) *Let $\gamma_k \geq 0$ and let C_k be measurable sets, $k = 1, \dots, n$. Then the function f defined by*

$$f = \sum_{k=1}^{n} \gamma_k \mathbb{1}_{C_k}$$

is a simple function and

$$\int_\Omega f d\mu = \sum_{k=1}^{n} \gamma_k \mu(C_k).$$

Definition 56 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let the extended real-valued function $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable and nonnegative. Then the integral is defined by*

$$\int_\Omega f\,d\mu := \left\{ \int_\Omega g\,d\mu \,:\, 0 \leq g \leq f,\, g \text{ simple} \right\} \in [0, \infty]$$

Remark: It is clear that if $(\Omega, \mathcal{A}, \mu)$ is a measure space and $E \in \mathcal{A}$, then

$$\int_\Omega \mathbb{1}_E d\mu = \mu(E)$$

Corollary 57 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let the extended real-valued function $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable and nonnegative. Then the integral of f with respect to μ over $A \in \mathcal{A}$ is*

$$\int_A f\,d\mu = \int_\Omega f \cdot \mathbb{1}_A\,d\mu.$$

Lemma 58 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let the extended real-valued function $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable and nonnegative. Let $f_n : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ be a sequence of nonnegative simple functions such that $f_n \uparrow f$. Then*

$$\int_\Omega f_n\,d\mu \uparrow \int_\Omega f\,d\mu.$$

Corollary 59 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $(\Omega, \mathcal{A}', \mu')$ be its completion. Let the extended real-valued function $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable and nonnegative. Then, in addition to f being measurable with respect to $(\Omega, \mathcal{A}')$, one has*

$$\int_\Omega f \, d\mu = \int_\Omega f \, d\mu',$$

and the equality is understood in $[0, \infty]$.

Proposition 60 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let the extended real-valued functions $f, g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable and nonnegative and let $a \geq 0$.*

- $\int_\Omega \alpha f \, d\mu = \alpha \int_\Omega f \, d\mu$;
- $\int_\Omega (f + g) \, d\mu = \int_\Omega f \, d\mu + \int_\Omega g \, d\mu$;
- $f \leq g \Rightarrow \int_\Omega f \, d\mu \leq \int_\Omega g \, d\mu$

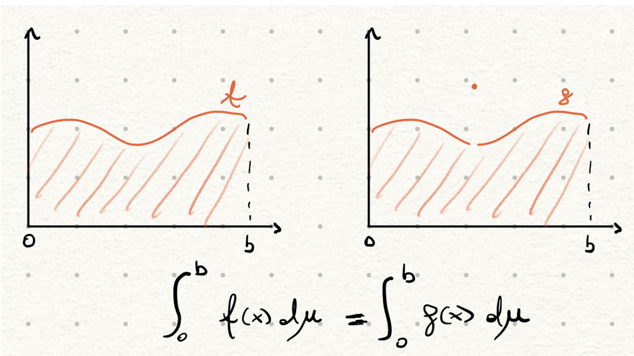
Corollary 61 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let the extended real-valued function $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable and nonnegative. Then*

$$f = 0 \; a.e \quad \Leftrightarrow \quad \int_\Omega f \, d\mu = 0$$

Corollary 62 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let the extended real-valued functions $f, g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable and nonnegative. Then*

$$f = g \quad (\mu-) a.e \quad \Leftrightarrow \quad \int_\Omega f \, d\mu = \int_\Omega g \, d\mu$$

Remark: What is means is that if we have a non continuous function, and the set containing all the points of discontinuity have measure zero, then the integral is just the volume described by the function continuous version of the function. Example, take the following one dimensional case,



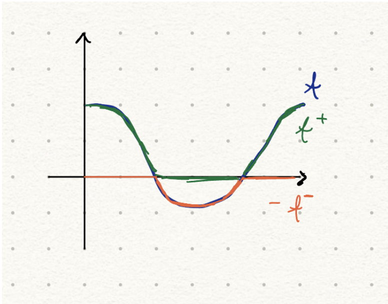
since $\mu(\{x \in \Omega : f(x) \neq g(x)\}) = 0$ then the integral are equal.

Corollary 63 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let the extended real-valued function $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable and nonnegative. Then*

$$\int_\Omega f \, d\mu < \infty \quad \Rightarrow \quad \mu(\{\omega \in \Omega : f(\omega) = \infty\}) = 0.$$

6.2 Integrable functions

Theorem 64 *An extended measurable real-valued function $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ can be written as the composition of the nonnegative measurable function f^+ and f^-.*



Definition 65 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let the extended real-valued function $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable. Then the integral is defined by*

$$\int_\Omega f \, d\mu = \int_\Omega f^+ \, d\mu - \int_\Omega f^- \, d\mu,$$

if at least one of the integrals in the rhs is finite. The function f is said to be inegrable if both integrals in the rhs are finite or, equivalently, if

$$\int_\Omega |f| \, d\mu < \infty.$$

Proposition 66 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let the extended real-valued functions $f, g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be integrable, and let $a \in \mathbb{R}$. Then αf and $f + g$ are also integrable and*

- $\int_\Omega \alpha f \, d\mu = \alpha \int_\Omega f \, d\mu$;
- $\int_\Omega (f + g) \, d\mu = \int_\Omega f \, d\mu + \int_\Omega g \, d\mu$;
- $f \leq g \Rightarrow \int_\Omega f \, d\mu \leq \int_\Omega g \, d\mu$;
- $| \int_\Omega f \, d\mu| \leq \int_\Omega |f| \, d\mu$.

Proposition 67 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let the extended real-valued functions $f, g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable. If f is also integrable and $f = g$ a.e. Then g is integral and $\int_\Omega g \, d\mu = \int_\Omega f \, d\mu$.*

Corollary 68 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let the extended real-valued function $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be integrable. Then*

$$\mu(\{\omega \in \Omega \, : \, |f(\omega)| = \infty\}) = 0,$$

and there exists an integrable real-valued function $g : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ such that

$$g = f \text{ a.e.} \quad \text{and} \quad \int_\Omega g \, d\mu = \int_\Omega f \, d\mu.$$

Theorem 69 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $(\Omega, \mathcal{A}', \mu')$ be its completion. Then if f is an extended integrable real-valued function from Ω to $\overline{\mathbb{R}}$, then f is also integrable with respect to $(\Omega, \mathcal{A}', \mu')$.*

6.2.1 Integration with respect to Lebesgue measure

Remark: Any continuos functions $f : \mathbb{R}^d \to \mathbb{R}$ is Borel measurable, and hence Lebesgue measurable.

Let cinsider the case $d = 1$. Let $f : [a, b] \to \mathbb{R}$ ve continuous and $a \leq b$. The set $[a, b]$ is provided with the Lebesgue σ-algebra of $\mathbb{R}$. Then, the integral $\int_{[a,b]} f\, d\mu$ is usually denoted by

$$\int_a^b f(t)\, d\mu(t)\, quad\text{or} \quad \int_a^b f(t)dt.$$

Lemma 70 *Let the real-valued function $f : [a, b] \to \mathbb{R}$ be continuous, then f is Lebesgue integrable.*

- *If the function $F : [a, b] \to \mathbb{R}$ is defined by*

  $$F(x) = \int_a^x f(t)\, dt, \quad x \in [a, b],$$

  then F is continuously differentiable on $[a, b]$ and $F' = f$.

if the function $F : [a, b] \to \mathbb{R}$ is continuously differentiable on $[a, b]$ and $F' = f$, then

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

Corollary 71 *Let the real-valued function $\gamma : [a, b] \to \mathbb{R}$ be continuously differentiable with $\gamma' > 0$, and let the real valued function $f'[\gamma(a), \gamma(b)] \to \mathbb{R}$ be continuous. Then $(f \circ \gamma)\gamma'$ is Lebesgue integrable on $[a, b]$ and*

$$\int_a^b (f \circ \gamma)(x)\gamma'(x)\, dx = \int_{\gamma(a)}^{\gamma(b)} f(y)\, dy.$$

7 Convergence Theorems

Theorem 72 ***Monotone Convergence Theorem*** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let f_n, $n \in \mathbb{N}$, and f be nonnegative measurable extended real-valued functions on Ω, such that*

$$f_n(\omega) \to f(\omega), \quad f_n(\omega) \leq f_{n+1}(\omega), \quad \omega \in \Omega.$$

Then

$$\lim_{n\to\infty} \int_\Omega f_n\, d\mu = \int_\Omega f\, d\mu.$$

Lemma 73 *Let $f : (a, b] \to \mathbb{R}$ or $f : [a, b) \to \mathbb{R}$ be nonnegative and continuous functions with primitive F. Then*

$$\int_{(a,b]} f\, d\mu = \lim_{\alpha \searrow a} [F(b) - F(\alpha)] \quad or \quad \int_{[a,b)} f\, d\mu = \lim_{\beta \nearrow b} [F(\beta) - F(a)],$$

respectively.

Theorem 74 ***Fatou's lemma*** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let f_n, $n \in \mathbb{N}$ be nonnegative measurable extended real-valued functions on Ω, then*

$$\int_\Omega (\liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int_\Omega f_n \, d\mu.$$

Theorem 75 ***Dominated convergence theorem*** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let f_n, $n \in \mathbb{N}$, and f be nonnegative measurable extended real-valued functions on Ω, and let $g : \Omega \to [0, \infty]$ be integrable extended real-valued function, such that*

$$f_n(\omega) \to f(\omega), \quad |f_n(\omega)| \le g(\omega), \quad \text{for almost all } \omega \in \Omega.$$

Then f_n and f are integrable and

$$\lim_{n\to\infty} \int_\Omega |f_n - f| \, d\mu = 0, \quad \lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega f \, d\mu.$$

Proposition 76 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let f_n be nonnegative measurable extended real-valued functions on Ω, then*

$$\int_\Omega \left(\sum_{n=1}^\infty f_n \right) d\mu = \sum_{n=1}^\infty \int_\Omega f_n \, d\mu.$$

Corollary 77 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and assume $\Omega = \bigcup_{n=1}^\infty \Omega_n$ with disjoint sets $\Omega_n \in \mathcal{A}$. Let f be a nonnegative measurable extended real-valued function on Ω. Then*

$$\int_\Omega f \, d\mu = \sum_{n=1}^\infty \int_{\Omega_n} f \, d\mu.$$

8 Parameter dependent integrals

Theorem 78 *Let $(\Omega, \mathcal{A}, \mu)$ be a complete measure space and let $I \subset \mathbb{R}$ be an open interval with $x_0 \in I$. Let $f : I \times \Omega \to \mathbb{R}$ be a real-valued function such that*

- *for every $x \in I$ the function $\omega \mapsto f(x, \omega)$ is integrable;*
- *for almost all $\omega \in \Omega$ the function $x \mapsto f(x, \omega)$ is continuous at $x_0 \in I$;*
- *there exits an integrable real-valued function $g : \Omega \to \mathbb{R}$ such that for every $x \in I$*

  $$|f(x, \omega)| \le |g(x)|$$

  holds almost everywhere on Ω.

Then the real-valued function $F : I \to \mathbb{R}$ defined by

$$F(x) = \int_\Omega f(x, \omega) \, d\mu(\omega), \quad x \in I,$$

is continuous at $x_0 \in I$.

Theorem 79 *Let $(\Omega, \mathcal{A}, \mu)$ be a complete measure space and let $I \subset \mathbb{R}$ be an open interval with $x_0 \in I$. Let $f : I \times \Omega \to \mathbb{R}$ be a real-valued function such that*

- *for every $x \in I$ the function $\omega \mapsto f(x, \omega)$ is integrable;*
- *for almost all $\omega \in \Omega$ the function $x \mapsto f(x, \omega)$ is differentiable at $x_0 \in I$;*
- *there exits an integrable real-valued function $g : \Omega \to \mathbb{R}$ such that for every $x \in I$*

$$|\partial_x f(x, \omega)| \leq |g(x)|$$

holds almost everywhere on Ω.

Then the real-valued function $F : I \to \mathbb{R}$ defined by

$$F(x) = \int_\Omega f(x, \omega)\, d\mu(\omega), \quad x \in I,$$

is differentiable at $x_0 \in I$ and

$$F'(x_0) = \int_\Omega \partial_x|_{x_0} f(x, \omega)\, d\mu(\omega).$$

9 Density and transformations of measures

Theorem 80 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let the extended real-valued function $h : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be nonnegative and measurable. Then λ defined by*

$$\lambda(E) = \int_E h\, d\mu, \quad E \in \mathcal{A},$$

is a measure on $\mathcal{A}$. Let $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be a measurable extended real-valued function. Then the following statements hold:

- *If the function f is nonnegative, then*

$$\int_\Omega f\, d\lambda = \int_\Omega fh\, d\mu.$$

- *The function f is integrable with respect to λ if and only if the function fh is integrable with respect to μ, in which case the previous equation holds.*

Remark: The function h is sometimes called the density of λ with respect to μ.

As already partially covered in the section Random Variables, we can have an induced measure.

Theorem 81 *Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be measurable spaces and assume that $\gamma : (\Omega, \mathcal{A}) \to (\Omega', \mathcal{A}')$ is a measurable mapping. If μ is a measure on $(\Omega, \mathcal{A})$, then ν defined by*

$$\nu(B) = \mu(\gamma^{-1}(B)), \quad B \in \mathcal{A}',$$

is a measure on $(\Omega', \mathcal{A}')$. Let the extended real-valued function $f : (\Omega', \mathcal{A}') \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable. Then the following statements hold.

- *If the function f is nonnegative, then*

$$\int_{\Omega'} f\, d\nu = \int_{\Omega} (f \circ \gamma)\, d\mu.$$

- *The function f is integrable with respect to ν if and only if the function $f \circ \gamma$ is integrable with respect to μ, in which case the previous equation holds.*

10 Product measures

Definition 82 *Let $\mathcal{A}_1$ be a σ-algebra on Ω_1 and let $\mathcal{A}_2$ be a σ-algebra on Ω_2. The σ-algebra on Ω generated by*

$$\mathcal{G} := \{\mathcal{A}_1 \times \mathcal{A}_2 \,:\, A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\},$$

is called the product σ-algebra $\mathcal{A} = \mathcal{A}_1 \otimes A_2$ on $\Omega_1 \otimes \Omega_2$, so that $\mathcal{A}_1 \otimes \mathcal{A}_2 = \sigma(\mathcal{G})$.

Remark: Note that for all $A_1, B_1 \in \mathcal{A}_1$ and $A_2, B_2 \in \mathcal{A}_2$:

$$(A_1 \times A_2) \cap (B_1 \times B_2) = (A_1 \cap B_1) \times (A_2 \cap B_2)$$

and that $A_1 \cap B_1 \in \mathcal{A}_1$ and $A_2 \cap B_2 \in \mathcal{A}_2$.

Definition 83 *Let A be a subset of $\Omega = \Omega_1 \times \Omega_2$. For $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ the sections A_{ω_1} and A^{ω_2} are defined as subsets of Ω_2 and Ω_1 respectively, by*

$$A_{\omega_1} = \{\omega_2 \in \Omega_2 \,:\, (\omega_1, \omega_2) \in \Omega\},$$

and

$$A^{\omega_2} = \{\omega_1 \in \Omega_1 \,:\, (\omega_1, \omega_2) \in \Omega\}.$$

Proposition 84 *Let $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$. Then*

1. $\omega_1 \in \Omega_1 \Rightarrow A_{\omega_1} \in \mathcal{A}_2$
2. $\omega_2 \in \Omega_2 \Rightarrow A^{\omega_2} \in \mathcal{A}_1$

Definition 85 *Let $f : \Omega_1 \times \Omega_2 \to \overline{\mathbb{R}}$ be an extended real-valued function. For $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ the sections $f_{\omega_1} : \Omega_2 \to \overline{\mathbb{R}}$ and $f^{\omega_2} : \Omega_1 \to \overline{\mathbb{R}}$ are defined by*

$$f_{\omega_1}(\omega_2) = f(\omega_1, \omega_2), \quad \omega_2 \in \Omega_2,$$

and

$$f^{\omega_2}(\omega_1) = f(\omega_1, \omega_2), \quad \omega_1 \in \Omega_1.$$

Proposition 86 *Let $f : (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_1) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be an extended real-valued measurable function. Then*

1. *for $\omega_1 \in \Omega_1$ the function $f_{\omega_1} : (\Omega_2, \mathcal{A}_2) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ is measurable;*
2. *for $\omega_2 \in \Omega_2$ the function $f^{\omega_2} : (\Omega_1, \mathcal{A}_1) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ is measurable.*

Remark: If $f : \Omega_1 \times \Omega_2 \to \overline{\mathbb{R}}$ then it is straiforward to see that

$$(f^+)_{\omega_1} = (f_{\omega_1})^+, \quad (f^-)_{\omega_1} = (f_{\omega_1})^-$$

Furthermore if we have two measure set $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$, and we let $A = A_1 \times A_2 \in \mathcal{G}$, then

$$\begin{aligned} \mu_2(A_{\omega_1}) &= \mu_2(A_2)\mathbb{1}_{A_1}(\omega_1), \quad \omega_1 \in \Omega_1 \\ \mu_1(A^{\omega_2}) &= \mu_1(A_1)\mathbb{1}_{A_2}(\omega_2), \quad \omega_2 \in \Omega_2 \end{aligned}$$

Proposition 87 *Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure space and let $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$. Then*

1. *the function $\omega_1 \mapsto \mu_2(A_{\omega_1})$ is measurable with respect to A_1;*

2. *the function $\omega_2 \mapsto \mu_1(A^{\omega_2})$ is measurable with respect to A_2.*

Theorem 88 *Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure space and let $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ be the corresponding measurable product space. Then*

$$\mu(A) = \int_{\Omega_1} \mu_2(A_{\omega_1})\mu_1(d\omega_1) = \int_{\Omega_2} \mu_1(A^{\omega_2})\mu_2(d\omega_2), \quad A \in \mathcal{A}_1 \otimes \mathcal{A}_2,$$

defines a σ-finite measure $\mu = \mu_1 \otimes \mu_2$ on $\mathcal{A}_1 \otimes \mathcal{A}_2$. Moreover $\mu = \mu_1 \otimes \mu_2$ is the only measure on $\mathcal{A}_1 \otimes \mathcal{A}_2$ which satisfies

$$\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2), \quad A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2.$$

Corollary 89 *Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure space with product measure $\mu_1 \otimes \mu_2$, and let $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$. Then the following statements are equivalent:*

- *$(\mu_1 \otimes \mu_2)(A) = 0$;*
- *$\mu_2(A_{\omega_1}) = 0$ for μ_1-almost all $\omega_1\Omega_1$;*
- *$\mu_1(A^{\omega_2}) = 0$ for μ_2-almost all $\omega_2\Omega_2$*

10.1 Fubuni-Tonelli Theorems

Theorem 90 ***Fubini-Tonelli*** *Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure space. Assume that $f : (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ us a measurable nonnegative extended real-valued function. Then*

1. *the function $\omega_1 \mapsto \int_{\Omega_2} f_{\omega_1} d\mu_2$ is nonnegative and $\mathcal{A}_1$-measurable;*

2. *the function $\omega_2 \mapsto \int_{\Omega_1} f^{\omega_2} d\mu_1$ is nonnegative and $\mathcal{A}_2$-measurable.*

Moreover,

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} f d(\mu_1 \otimes \mu_2) &= \int_{\Omega_1} \left(\int_{\Omega_2} f_{\omega_1} d\mu_2 \right) d\mu_1(\omega_1) \\ &= \int_{\Omega_2} \left(\int_{\Omega_1} f^{\omega_2} d\mu_1 \right) d\mu_2(\omega_2) \end{aligned}$$

Theorem 91 ***Fubini-Tonelli*** *Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure space. Assume that $f : (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ us a measurable nonnegative extended real-valued function. Then the set*

$$F_1 = \{\omega_1 \in \Omega_1 \, : \, f_{\omega_1} \text{ not integrable w.r.t } \mu_2\}$$

has μ_1-measure 0, and the set

$$F_2 = \{\omega_2 \in \Omega_2 \, : \, f^{\omega_2} \text{ not integrable w.r.t } \mu_1\}$$

has μ_2-measure 0. Moreover,

$$\begin{aligned}\int_{\Omega_1 \times \Omega_2} f d(\mu_1 \otimes \mu_2) &= \int_{\Omega_1 \setminus F_1} \left(\int_{\Omega_2} f_{\omega_1} d\mu_2 \right) d\mu_1(\omega_1) \\ &= \int_{\Omega_2 \setminus F_2} \left(\int_{\Omega_1} f^{\omega_2} d\mu_1 \right) d\mu_2(\omega_2)\end{aligned}$$

10.2 Product measures on $\mathbb{R}^p \times \mathbb{R}^q$

Proposition 92 *The product of Borel σ-algebra $\mathcal{B}^p \otimes \mathcal{B}^q$ satisfies*

$$\mathcal{B}^p \otimes \mathcal{B}^q = \mathcal{B}^{p+q}.$$

Moreover, the product measure $\mu_p \otimes \mu_q$ on $\mathcal{B}^p \otimes \mathcal{B}^q$ is well-defined and satisfies

$$\mu_p \otimes \mu_q = \mu_{p+q}$$

where μ_{p+q} is Lebesgue measure on $\mathcal{B}^{p+q}$.

11 Space of Integrable functions

This section used concept from function analysis. To optimize this summary I will omit the definition of semi-norm, Banach spaces and the relatives theorems of convergences of sequences.

Definition 93 *Let $\Omega, \mathcal{A}, \mu)$ be a measure space. The integrable functions on Ω form a linear space $\mathcal{L}^1(\Omega, \mathcal{A}, \mu)$ and clearly $\|f\| = \int_\Omega |f| d\mu$ provides a semi-norm on that space.*

Lemma 94 *Let $\alpha, \beta \in \mathbb{R}$ be nonnegative and let $1 \leq p < \infty$ and $1/p + 1/q = 1$. Then*

$$\alpha^{1/p}\beta^{1/q} \leq \frac{\alpha}{p} + \frac{\beta}{q},$$

and

$$\left(\frac{\alpha + \beta}{2}\right)^p \leq \frac{1}{2}(\alpha^p + \beta^p)$$

Definition 95 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $1 \leq p \leq \infty$. For $p < \infty$ the space $\mathcal{L}^p(\Omega)$ is the set of all measurable complex-valued functions f for which*

$$\int_\Omega |f|^p d\mu < \infty.$$

Furthermore, one defines for $f \in \mathcal{L}^p(\Omega)$

$$\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{1/p}.$$

For $p = \infty$ the space $\mathcal{L}^\infty(\Omega)$ is the set of all measurable complect-valued functions f for which there exists $c \geq 0$ such that

$$|f(\omega)| \leq c \quad a.e.$$

Furthermore, one defines for $f \in \mathcal{L}^\infty(\Omega)$

$$\|f\|_\infty = \inf\{c \geq 0 \,:\, |f(\omega)| \leq c\, a.e.\}.$$

Lemma 96 *The space $\mathcal{L}^p(\Omega)$, $1 \leq p \leq \infty$, is a linear space.*

Theorem 97 ***Holder's Inequality*** *Let $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. If $f \in \mathcal{L}^p(\Omega)$ and $g \in \mathcal{L}^q(\Omega)$, then $fg \in \mathcal{L}^1(\Omega)$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

In particular, for $p = 2$ and $q = 2$ this is the Cauchy-Schwarz inequality.

Corollary 98 *Let $1 \leq p \leq \infty$ and assume that $\mu(\Omega) = 1$. If $f \in \mathcal{L}^p(\Omega)$ then $f \in \mathcal{L}^1(\Omega)$ and*

$$\|f\|_1 \leq \|f\|_p.$$

Theorem 99 ***Mikowski's inequality*** *Let $1 \leq p \leq \infty$. The space $\mathcal{L}^p(\Omega)$ with $\|\cdot\|_p$ is a semi-normed linear space. In particular for $f, g \in \mathcal{L}^p(\Omega)$*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

A function $f \in \mathcal{L}^p(\Omega)$ has $\|f\|_p = 0$ if and only if $f = 0$ almost everywhere.

11.1 Completeness

Theorem 100 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $1 \leq p < \infty$. Let $f_n \in \mathcal{L}^p(\Omega)$, $n \in \mathbb{N}$, and assume that the series $sum_{k=1}^\infty f_k$ converges absolutely. Then the series $sum_{k=1}^\infty f_k$ converges in $\mathcal{L}^p(\Omega)$, i.e., there exists a function $f \in \mathcal{L}^p(\Omega)$, such that*

$$\sum_{k=1}^\infty f_k = f,$$

where the series converges in $\mathcal{L}^p(\Omega)$. In addition, the series converges pointwise almost everywhere.

Theorem 101 *Let $1 \leq p \leq \infty$. The semi-normed linear space $\mathcal{L}^p(\Omega)$ is complete.*

Corollary 102 *Let $1 \leq p < \infty$. Each Cauchy sequence in $\mathcal{L}^p(\Omega)$ contains a subsequences which converges pointwise almost everywhere.*

Let $\mathcal{N}(\mu)$ be the closed linear subspace of all elements in $\mathcal{L}^p(\Omega, \mu)$ whose semi-norm is zero.

Theorem 103 *The natural embedding from $\mathcal{L}^p(\Omega, \mu)/\mathcal{N}(\mu)$ into $\mathcal{L}^p(\Omega, \mu')/\mathcal{N}(\mu')$ is surjective.*

11.2 Dense linear subsets

Theorem 104 *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $1 \leq p \leq \infty$. The simple functions which belongs to $\mathcal{L}^p(\Omega)$ are dense in $\mathcal{L}^p(\Omega)$.*

Let $C_c(\mathbb{R}^d)$ be the collection of all complex-valued functions on $\mathbb{R}^d$ which are continuous and which have compact support. It is clear that $C_c(\mathbb{R}^d)$ is a linear space and that it is contained in $\mathcal{L}^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$.

Theorem 105 *The space $C_c(\mathbb{R}^d)$ is dense in $\mathcal{L}^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$.*

Corollary 106 *Let $f \in \mathcal{L}^p(\mathbb{R}^d)$, $1 \leq p < \infty$, then*

$$\lim_{x \to 0} \int_{\mathbb{R}^d} |f(x+t) - f(t)|^p d\mu(t) = 0$$

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